A Hodge dual for soldered bundles

2009 J. Phys. A: Math. Theor. 42 035402

(http://iopscience.iop.org/1751-8121/42/3/035402)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 035402 (8pp) doi:10.1088/1751-8113/42/3/035402

A Hodge dual for soldered bundles

Tiago Gribl Lucas and J G Pereira

Instituto de Física Teórica, Universidade Estadual Paulista, Rua Pamplona 145, 01405-900 São Paulo, Brazil

Received 19 October 2008
Published 9 December 2008
Online at stacks.iop.org/JPhysA/42/035402

Abstract

In order to account for all possible contractions allowed by the presence of the solder form, a generalized Hodge dual is defined for the case of soldered bundles. Although for curvature the generalized dual coincides with the usual one, for torsion it gives a completely new dual definition. Starting from the standard form of a gauge Lagrangian for the translation group, the generalized Hodge dual yields precisely the Lagrangian of the teleparallel equivalent of general relativity, and consequently also the Einstein–Hilbert Lagrangian of general relativity.

PACS numbers: 02.40.Hw, 04.20.Cv, 11.15.−q

1. Introduction

The geometrical setting of any gravitational theory is the tangent bundle, a natural construction always present in spacetime. According to this structure, at each point of spacetime—the base space of the bundle—there is a tangent space attached to it—the fiber of the bundle—on which the gauge group acts[1]. Differently from *internal* bundles of the Yang–Mills-type gauge theories, spacetime-rooted bundles, as for example the tangent bundle, have a quite peculiar property: the presence of the solder form, whose components are the tetrad field [1]. For this reason, they are called soldered bundles. An immediate consequence of this property is that the connections living in these bundles will present, in addition to curvature, also torsion. This is the case, for example, of the Levi-Civita connection of general relativity, which has vanishing torsion[2].

[1] We use the Greek alphabet ($\mu, \nu, \rho, \ldots = 0, 1, 2, 3$) to denote indices related to spacetime, and the Latin alphabet ($a, b, c, \ldots = 0, 1, 2, 3$) to denote algebraic indices related to the tangent space, assumed to be a Minkowski spacetime with the metric. $\eta_{ab} = \mathrm{diag}(+1, -1, -1, -1)$.

[2] We remark that the presence of a vanishing torsion is completely different from the absence of torsion, which is the case of the non-soldered bundles of internal (or Yang–Mills) gauge theories.

We denote the spacetime coordinates by x^μ, whereas the tangent space coordinates will be denoted by x^a. Since x^a are functions of x^μ, we can define coordinate basis for vector fields and their duals in the form

$$\partial_a = (\partial^\mu x_a)\partial_\mu \qquad \text{and} \qquad \partial^a = (\partial_\mu x^a)\,\mathrm{d}x^\mu. \tag{1}$$

In these expressions, $\partial_\mu x^a$ is a trivial—that is, holonomic—tetrad, with $\partial^\mu x_a$ its inverse. A nontrivial tetrad field, on the other hand, defines naturally a non-coordinate basis for vector fields and their duals,

$$h_a = h_a{}^\mu \partial_\mu \qquad \text{and} \qquad h^a = h^a{}_\mu\,\mathrm{d}x^\mu. \tag{2}$$

These basis are non-holonomic,

$$[h_c, h_d] = f^a{}_{cd} h_a, \tag{3}$$

with

$$f^a{}_{cd} = h_c{}^\mu h_d{}^\nu \big(\partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu\big) \tag{4}$$

the coefficient of anholonomy. A fundamental property of soldered bundles is that the spacetime (external) and the tangent–space (internal) metrics are related by

$$g_{\mu\nu} = \eta_{ab} h^a{}_\mu h^b{}_\nu. \tag{5}$$

A spin connection A_μ is a connection assuming values in the Lie algebra of the Lorentz group,

$$A_\mu = \tfrac{1}{2} A^{ab}{}_\mu S_{ab}, \tag{6}$$

with S_{ab} a given representation of the Lorentz generators. The corresponding covariant derivative is the Fock–Ivanenko operator [2, 3]:

$$\mathcal{D}_\mu = \partial_\mu - \frac{\mathrm{i}}{2} A^{ab}{}_\mu S_{ab}. \tag{7}$$

Acting on a Lorentz vector field ϕ^a, for example, S_{ab} is the matrix [4]

$$(S_{ab})^c{}_d = \mathrm{i}\big(\delta_a{}^c \eta_{bd} - \delta_b{}^c \eta_{ad}\big),$$

and consequently

$$\mathcal{D}_\mu \phi^a = \partial_\mu \phi^a + A^a{}_{b\mu} \phi^b. \tag{8}$$

The spacetime linear connection $\Gamma^\rho{}_{\nu\mu}$ corresponding to $A^a{}_{b\mu}$ is

$$\Gamma^\rho{}_{\nu\mu} = h_a{}^\rho \partial_\mu h^a{}_\nu + h_a{}^\rho A^a{}_{b\mu} h^b{}_\nu \equiv h_a{}^\rho \mathcal{D}_\mu h^a{}_\nu. \tag{9}$$

The inverse relation is

$$A^a{}_{b\mu} = h^a{}_\nu \partial_\mu h_b{}^\nu + h^a{}_\nu \Gamma^\nu{}_{\rho\mu} h_b{}^\rho \equiv h^a{}_\nu \nabla_\mu h_b{}^\nu. \tag{10}$$

Equations (9) and (10) are different ways of expressing the property that the total covariant derivative—that is, with connection term for both indices—of the tetrad vanishes identically:

$$\partial_\mu h^a{}_\nu - \Gamma^\rho{}_{\nu\mu} h^a{}_\rho + A^a{}_{b\mu} h^b{}_\nu = 0. \tag{11}$$

From a formal point of view, curvature and torsion are properties of connections. This becomes evident if we observe that many connections, with different curvature and torsion, are allowed to exist in the very same metric spacetime [5]. Given a connection $A^a{}_{b\mu}$, its curvature and the torsion are defined respectively by

$$R^a{}_{b\nu\mu} = \partial_\nu A^a{}_{b\mu} - \partial_\mu A^a{}_{b\nu} + A^a{}_{e\nu} A^e{}_{b\mu} - A^a{}_{e\mu} A^e{}_{b\nu} \tag{12}$$

and

$$T^a{}_{\nu\mu} = \partial_\nu h^a{}_\mu - \partial_\mu h^a{}_\nu + A^a{}_{e\nu} h^e{}_\mu - A^a{}_{e\mu} h^e{}_\nu. \tag{13}$$

Using relation (10), they can be expressed in a purely spacetime form

$$R^{\rho}{}_{\lambda\nu\mu} = \partial_\nu \Gamma^{\rho}{}_{\lambda\mu} - \partial_\mu \Gamma^{\rho}{}_{\lambda\nu} + \Gamma^{\rho}{}_{\eta\nu}\Gamma^{\eta}{}_{\lambda\mu} - \Gamma^{\rho}{}_{\eta\mu}\Gamma^{\eta}{}_{\lambda\nu} \tag{14}$$

and

$$T^{\rho}{}_{\nu\mu} = \Gamma^{\rho}{}_{\mu\nu} - \Gamma^{\rho}{}_{\nu\mu}. \tag{15}$$

The connection coefficients can be decomposed according to[3]

$$\Gamma^{\rho}{}_{\mu\nu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\mu\nu} + K^{\rho}{}_{\mu\nu}, \tag{16}$$

where

$$\overset{\circ}{\Gamma}{}^{\sigma}{}_{\mu\nu} = \tfrac{1}{2} g^{\sigma\rho}(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \tag{17}$$

is the Levi-Civita connection of general relativity, and

$$K^{\rho}{}_{\mu\nu} = \tfrac{1}{2}\left(T_\nu{}^{\rho}{}_{\mu} + T_\mu{}^{\rho}{}_{\nu} - T^{\rho}{}_{\mu\nu}\right) \tag{18}$$

is the contortion tensor. Using relation (9), the decomposition (16) can be rewritten as

$$A^{c}{}_{a\nu} = \overset{\circ}{A}{}^{c}{}_{a\nu} + K^{c}{}_{a\nu}, \tag{19}$$

where $\overset{\circ}{A}{}^{c}{}_{a\nu}$ is the Ricci coefficient of rotation, the spin connection of general relativity.

2. Dual operation for soldered bundles

2.1. General notions

Let Ω^p be the space of p-forms on an n-dimensional manifold M. Since the vector spaces Ω^p and Ω^{n-p} have the same dimension, they are isomorphic. The choice of an orientation and the presence of a metric on TM then enables us to single out a unique isomorphism, the so-called Hodge dual [6]. For a p-form $\alpha^p \in \Omega^p$,

$$\alpha^p = \frac{1}{p!}\alpha_{\mu_1\ldots\mu_p}\omega^{\mu_1}\wedge\cdots\wedge\omega^{\mu_p}, \tag{20}$$

its Hodge dual is the $(n-p)$-form $\star\alpha^p \in \Omega^{n-p}$ defined by

$$\star\,\alpha^p = \frac{h}{(n-p)!\,p!}\epsilon_{\mu_1\mu_2\ldots\mu_n}\alpha^{\mu_1\ldots\mu_p}\omega^{\mu_{p+1}}\wedge\cdots\wedge\omega^{\mu_n}, \tag{21}$$

where we have used the identification $h = \sqrt{-g}$, with $h = \det\left(h^a{}_\mu\right)$ and $g = \det(g_{\mu\nu})$. The operator $\star$ satisfies the property

$$\star\star\alpha^p = (-1)^{p(n-p)+(n-s)/2}\alpha^p, \tag{22}$$

where s is the signature of the spacetime metric. Its inverse is given by

$$\star^{-1} = (-1)^{p(n-p)+(n-s)/2}\,\star. \tag{23}$$

[3] All magnitudes related to general relativity will be denoted by an over '∘'.

2.2. The case of non-soldered bundles

For non-soldered bundles, the dual operator can be defined in a straightforward way to act on vector-valued p-forms. Let β be a vector-valued p-form on the n-dimensional base space M, taking values on a vector space F. Its dual is the vector-valued $(n-p)$-form

$$\star\,\beta^p = \frac{h}{(n-p)!p!}\epsilon_{\mu_1\mu_2\ldots\mu_n} e_i \beta^{i\mu_1\cdots\mu_p}\omega^{\mu_{p+1}} \wedge \cdots \wedge \omega^{\mu_n}, \tag{24}$$

where the set $\{e_i\}$ is a basis for the vector space F. In this case, the components $\beta^{i\mu_1\cdots\mu_p}$ have also an internal space index i, which is not related to the external indices μ_i. The property (22) is of course still valid. As an example, let us consider the Yang–Mills field strength $F^A{}_{\mu\nu}$ in a four-dimensional spacetime. As the algebraic indices $(A, B, \ldots)$ are not related to the spacetime indices $(\mu, \nu, \ldots)$, the Hodge dual is defined by [7]

$$\star\, F^A{}_{\mu\nu} = \frac{h}{2}\epsilon_{\mu\nu\rho\sigma} F^{A\rho\sigma}. \tag{25}$$

2.3. The case of soldered bundles

The case of soldered bundles is quite different. Due to the presence of the solder form, internal and external indices can be transformed into each other, and this feature leads to the possibility of defining new dual operators, each one related to an inner product on Ω^p. The main requirement of these new definitions is that, since (22) is still valid for p-forms on soldered bundles, we want to make it true also for vector-valued p-forms. We consider next, in a four-dimensional spacetime, the specific cases of torsion and curvature.

2.3.1. Torsion. Differently from internal (non-soldered) gauge theories, whose dual is defined by equation (25), in soldered bundles algebraic and spacetime indices can be transformed into each other through the use of the tetrad field. This property opens up the possibility of new contractions in relation to the usual definition (25). There are basically two different kinds of terms that can be taken into account when defining a generalized dual torsion. They are given by

$$\star\, T^\lambda{}_{\mu\nu} = h\epsilon_{\mu\nu\rho\sigma}\left[a\left(\tfrac{1}{2}T^{\lambda\rho\sigma} + T^{\rho\lambda\sigma}\right) + bT^{\theta\rho}{}_\theta g^{\lambda\sigma}\right], \tag{26}$$

with a, b constant coefficients[4]. The factor '1/2' in the first term is necessary to remove equivalent terms of the summation. Now, in a four-dimensional spacetime with metric signature $s = 2$, the dual torsion must satisfy the relation

$$\star\star T^\rho{}_{\mu\nu} = -T^\rho{}_{\mu\nu}. \tag{27}$$

This condition yields the following algebraic system:

$$2a^2 - ab = 1 \tag{28}$$

$$2a^2 + ab = 0. \tag{29}$$

There are two solutions which differ by a global sign[5]

$$a = 1/2 \qquad b = -1 \tag{30}$$

[4] See the appendix for a demonstration that two coefficients suffice to define the generalized dual torsion.

[5] We remark that, if instead of the '+' sign in the middle term of (26) we had chosen a '−' sign, the algebraic system would become inconsistent.

and

$$a = -1/2 \qquad b = 1. \tag{31}$$

Since we are looking for a generalization of the usual expression (25), we choose the solution with $a > 0$. In this case, the generalized dual torsion reads

$$\star\, T^{\rho}{}_{\mu\nu} = h\epsilon_{\mu\nu\alpha\beta}\left(\tfrac{1}{4}T^{\rho\alpha\beta} + \tfrac{1}{2}T^{\alpha\rho\beta} - T^{\lambda\alpha}{}_{\lambda}g^{\rho\beta}\right). \tag{32}$$

Defining the tensor

$$S^{\rho\mu\nu} = -S^{\rho\nu\mu} := K^{\mu\nu\rho} - g^{\rho\nu}T^{\sigma\mu}{}_{\sigma} + g^{\rho\mu}T^{\sigma\nu}{}_{\sigma}, \tag{33}$$

the generalized Hodge dual torsion assumes the form

$$\star\, T^{\rho}{}_{\mu\nu} = \frac{h}{2}\epsilon_{\mu\nu\alpha\beta}S^{\rho\alpha\beta}. \tag{34}$$

We remark that solutions (30) and (31) are the only ones that make the dual torsion explicitly depend on the contortion tensor.

2.3.2. Curvature. Let us consider now the curvature tensor. Analogously to the torsion case, we define its generalized dual by taking into account all possible contractions,

$$\star R^{\alpha\beta}{}_{\mu\nu} = h\epsilon_{\mu\nu\rho\sigma}[aR^{\alpha\beta\rho\sigma} + b(R^{\alpha\rho\beta\sigma} - R^{\beta\rho\alpha\sigma}) + c(g^{\alpha\rho}R^{\beta\sigma} - g^{\beta\rho}R^{\alpha\sigma}) + dg^{\alpha\rho}g^{\beta\sigma}R], \tag{35}$$

with a, b, c, d constant coefficients. We remark that the anti-symmetry in α and β is necessary because the curvature 2-form takes values on the Lie algebra of the Lorentz group. By requiring that

$$\star\star R^{\alpha\beta}{}_{\mu\nu} = -R^{\alpha\beta}{}_{\mu\nu}, \tag{36}$$

we obtain a system of algebraic equations for a, b, c, d, whose unique solution is

$$a = 1/2 \qquad \text{and} \qquad b = c = d = 0. \tag{37}$$

This means that for curvature the generalized Hodge dual coincides with the usual definition, that is,

$$\star R^{\alpha\beta}{}_{\mu\nu} = \frac{h}{2}\epsilon_{\mu\nu\rho\sigma}R^{\alpha\beta\rho\sigma}. \tag{38}$$

3. An application: gravitational Lagrangian

Teleparallel gravity [8] is characterized by the vanishing of the spin connection[6]: $\overset{\bullet}{A}{}^{a}{}_{b\mu} = 0$. The curvature and torsion tensors in this case are given respectively by

$$\overset{\bullet}{R}{}^{a}{}_{b\nu\mu} = 0 \qquad \text{and} \qquad \overset{\bullet}{T}{}^{a}{}_{\nu\mu} = \partial_\nu h^{a}{}_{\mu} - \partial_\mu h^{a}{}_{\nu}. \tag{39}$$

Through a contraction with a tetrad, the torsion tensor assumes the form

$$\overset{\bullet}{T}{}^{\rho}{}_{\nu\mu} = \overset{\bullet}{\Gamma}{}^{\rho}{}_{\mu\nu} - \overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu}, \tag{40}$$

where

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} = h_{a}{}^{\rho}\partial_\mu h^{a}{}_{\nu} \tag{41}$$

is the Weitzenböck connection. It can be decomposed in the form

$$\overset{\bullet}{\Gamma}{}^{\rho}{}_{\nu\mu} = \overset{\circ}{\Gamma}{}^{\rho}{}_{\nu\mu} + \overset{\bullet}{K}{}^{\rho}{}_{\nu\mu}, \tag{42}$$

[6] All magnitudes related to teleparallel gravity will be denoted by an over '•'.

with $\overset{\bullet}{K}{}^{\rho}{}_{\nu\mu}$ the contortion of the Weitzenböck torsion.

Now, teleparallel gravity corresponds to a gauge theory for the translation group [9]. As such, its action is given by [10]

$$\overset{\bullet}{\mathcal{S}} = \frac{1}{ck} \int \eta_{ab} \overset{\bullet}{T}{}^a \wedge \star \overset{\bullet}{T}{}^b, \tag{43}$$

where $k = 16\pi G/c^4$ and

$$\overset{\bullet}{T}{}^a = \tfrac{1}{2} \overset{\bullet}{T}{}^a{}_{\mu\nu} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \qquad \text{and} \qquad \star \overset{\bullet}{T}{}^a = \tfrac{1}{2} \star \overset{\bullet}{T}{}^a{}_{\rho\sigma} \, \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma \tag{44}$$

are respectively the torsion 2-form and the corresponding dual form. Substituting these expressions into equation (43), it becomes

$$\overset{\bullet}{\mathcal{S}} = \frac{1}{4ck} \int \eta_{ab} \overset{\bullet}{T}{}^a{}_{\mu\nu} \star \overset{\bullet}{T}{}^b{}_{\rho\sigma} \, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma. \tag{45}$$

Using the identity

$$\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \wedge \mathrm{d}x^\sigma = -\epsilon^{\mu\nu\rho\sigma} h^2 \, \mathrm{d}^4x, \tag{46}$$

the action functional reduces to

$$\overset{\bullet}{\mathcal{S}} = -\frac{1}{4ck} \int \overset{\bullet}{T}{}_{\alpha\mu\nu} \star \overset{\bullet}{T}{}^\alpha{}_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma} h^2 \, \mathrm{d}^4x. \tag{47}$$

Using then the generalized dual definition (34), as well as the identity

$$\epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\rho\sigma} = -\frac{2}{h^2} \left(\delta^\mu_\alpha \delta^\nu_\beta - \delta^\nu_\alpha \delta^\mu_\beta \right), \tag{48}$$

we get

$$\overset{\bullet}{\mathcal{S}} = \frac{1}{2ck} \int \overset{\bullet}{T}{}_{\rho\mu\nu} \overset{\bullet}{S}{}^{\rho\mu\nu} h \, \mathrm{d}^4x. \tag{49}$$

This action yields the Lagrangian

$$\overset{\bullet}{\mathcal{L}} = \frac{h}{2k} \overset{\bullet}{T}{}_{\rho\mu\nu} \overset{\bullet}{S}{}^{\rho\mu\nu}, \tag{50}$$

which is precisely the Lagrangian of the teleparallel equivalent of general relativity [11]. Using equations (40) and (42), a straightforward calculation shows that it can be rewritten in the form

$$\overset{\bullet}{\mathcal{L}} = -\frac{h}{k} \overset{\circ}{R} - \partial_\mu \left(\frac{2h}{k} \overset{\bullet}{T}{}^{\nu\mu}{}_\nu \right). \tag{51}$$

Up to a divergence, therefore, the Lagrangian of a gauge theory for the translation group with the Hodge dual given by equation (32) yields the Einstein–Hilbert Lagrangian of general relativity. This shows the consistency—and actually the necessity—of the generalized Hodge dual definition (34).

4. Final remarks

For soldered bundles, the Hodge dual must be generalized in order to take into account all additional contractions allowed by the presence of the solder form. Although for curvature the generalized dual operation turns out to coincide with the usual one, for torsion it gives a completely new dual definition. The importance of this new definition can be verified by analyzing several aspects of gravitation. For example, starting from the standard Lagrangian of a gauge theory for the translation group, it naturally yields the Lagrangian of the teleparallel equivalent of general relativity, and consequently also the Einstein–Hilbert Lagrangian of general relativity. This is to say, it connects the Einstein–Hilbert Lagrangian with a typical gauge Lagrangian. It is important to remark that the generalized Hodge dual (34) has already been used previously [12], but it was guessed just to yield the desired result. Here we have shown that it can be obtained in a constructive way from first principles.

Acknowledgments

The authors would like to thank R Aldrovandi for useful discussions. They would also like to thank FAPESP, CAPES and CNPq for partial financial support.

Appendix. Torsion decomposition and the dual

Let us suppose that, instead of (26), the generalized dual torsion is defined by

$$\star T^{\lambda}{}_{\mu\nu} = h\epsilon_{\mu\nu\rho\sigma}\left(\alpha T^{\lambda\rho\sigma} + \beta T^{\rho\lambda\sigma} + \gamma T^{\theta\rho}{}_{\theta} g^{\lambda\sigma}\right), \tag{A.1}$$

with α, β, γ three constant coefficients. In terms of irreducible components under the global Lorentz group [13], the torsion can be written as

$$T_{\lambda\mu\nu} = \tfrac{2}{3}(t_{\lambda\mu\nu} - t_{\lambda\nu\mu}) + \tfrac{1}{3}(g_{\lambda\mu}v_\nu - g_{\lambda\nu}v_\mu) + \epsilon_{\lambda\mu\nu\rho}a^\rho, \tag{A.2}$$

where

$$v_\mu = T^\nu{}_{\nu\mu} \qquad \text{and} \qquad a^\mu = \tfrac{1}{6}\epsilon^{\mu\nu\rho\sigma}T_{\nu\rho\sigma} \tag{A.3}$$

are respectively the vector and axial vector parts, and

$$t_{\lambda\mu\nu} = \tfrac{1}{2}(T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \tfrac{1}{6}(g_{\nu\lambda}v_\mu + g_{\nu\mu}v_\lambda) - \tfrac{1}{3}g_{\lambda\mu}v_\nu \tag{A.4}$$

is a purely tensor part, that is, a tensor with vanishing vector and axial torsions. Using the generalized dual definition (A.1), a simple calculation shows that

$$\star v_\mu = -6h(\alpha - \beta)a_\mu \equiv Aha_\mu \tag{A.5}$$

and

$$\star a_\mu = \frac{1}{3h}(2\alpha + \beta + 3\gamma)v_\mu \equiv \frac{B}{h}v_\mu, \tag{A.6}$$

where A and B are two new parameters which, on account of the property (27), satisfy the relation $AB = -1$. In terms of the irreducible components, the generalized dual torsion is then found to be

$$\star T^\lambda{}_{\mu\nu} = h\left[\pm\frac{2}{3}\epsilon_{\mu\nu\alpha\beta}t^{\lambda\alpha\beta} + \frac{A}{3}\left(\delta^\lambda{}_\mu a_\nu - \delta^\lambda{}_\nu a_\mu\right) + \frac{B}{h^2}\epsilon^\lambda{}_{\mu\nu\rho}v^\rho\right]. \tag{A.7}$$

We see from this expression that two parameters suffice to define the generalized dual.

References

[1] Kobayashi S and Nomizu K 1963 *Foundations of Differential Geometry* (New York: Interscience)
[2] Fock V A and Ivanenko D 1929 *Z. Phys.* **54** 798
Fock V A 1929 *Z. Phys.* **57** 261
[3] Dirac P A M 1958 *Planck Festscrift* ed W Frank (Berlin: Deutscher Verlag der Wissenschaften)
[4] Ramond P 1989 *Field Theory: A Modern Primer* 2nd edn (Redwood: Addison-Wesley)
[5] Aldrovandi R and Pereira J G 1995 *An Introduction to Geometrical Physics* (Singapore: World Scientific)
[6] Baez J C and Muniain J P 1994 *Gauge Fields, Knots and Gravity* (Singapore: World Scientific)
[7] Frankel T 1977 *The Geometry of Physics* (Cambridge: Cambridge University Press)

[8] Arcos H I and Pereira J G 2004 *Int. J. Mod. Phys.* D **13** 2193 (arXiv:gr-qc/0501017)
[9] de Andrade V C, Guillen L C T and Pereira J G 2000 *Phys. Rev. Lett.* **84** 4533 (arXiv:gr-qc/0003100)
[10] Faddeev L D and Slavnov A A 1980 *Gauge Fields* (Reading: Benjamin-Cummings)
[11] Maluf J W 1994 *J. Math. Phys.* **35** 335
[12] Andrade V C, Barbosa A L and Pereira J G 2005 *Int. J. Mod. Phys.* D **14** 1635 (arXiv:gr-qc/0501037)
[13] Hayashi K and Bregman A 1973 *Ann. Phys., NY* **75** 562